

Soul Theorem and Soul Conjecture

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Structure of complete manifolds of nonnegative curvature

It is a central question in Riemannian geometry about the structure of complete manifolds whose sectional curvature is of a fixed sign. Before Cheeger and Gromoll, most attention had been paid to compact case, with an exception that is the work of Cohn-Vossen. He actually obtained the following result:

Theorem (Cohn-Vossen)

In dimension 2, a noncompact complete manifold of nonnegative curvature is either diffeomorphic to \mathbb{R}^2 or is flat.

Structure of complete manifolds of nonnegative curvature

In 1972, Cheeger and Colding generalized Cohn-Vossen's result as follows:

Theorem (Soul Theorem)

Let M be a complete noncompact Riemannian manifold with nonnegative sectional curvature. Then M contains a compact totally geodesic and totally convex embedding submanifold S without boundary, $0 \leq \dim S < \dim M$, such that M is diffeomorphic to the total space of the normal bundle of S in M .

Finally in their paper, they asked a natural question whether the result of Cohn-Vossen still holds for higher dimension. This is called the **Soul Conjecture**.

The Soul Conjecture

Suppose M is complete and noncompact with sectional curvature ≥ 0 , and there exists at least one point p such that the curvature is positive for all sections in T_pM . Then the soul of M is always a point, which implies that M is diffeomorphic to Euclidean space \mathbb{R}^n .

The Soul Conjecture

The soul conjecture holds if M has positive sectional curvature, due to an earlier result of Gromoll and Meyer. The case $n = 3$ was verified by Yu.D.Burago in 1979 and case $n = 4$ verified by V.B. Marenich in 1980. In 1992 and 1993, Marenich published an argument for analytic manifolds without dimensional restrictions, containing over 50 pages of computations.

Finally in 1994, G.Perelman presented an elegant proof of Soul Conjecture in 4 pages.

The proof depends mainly on two results: the Berger's version of Rauch comparison theorem and the existence of distance nonincreasing retraction of M onto S due to Sharafutdinov.

We still need the following results:

Lemma

A real value function on an interval whose upper left derivative is nowhere positive is a non-increasing function.

Lemma

Let M, N be complete Riemannian manifolds, $f : N \rightarrow M$ be an isometric embedding. Assume more that f is proper, i.e. the preimage of compact subset is compact. Then the normal exponential map is surjective.

Perelman's proof of Soul Conjecture

In fact, Perelman proved the following result:

Theorem (Perelman, 1994)

Let M be a complete noncompact Riemannian manifold of nonnegative sectional curvature, let S be a soul of M , and let $P : M \rightarrow S$ be a distance nonincreasing retraction. Then:

(A) For any $x \in S, \nu \in SN(S)$, we have

$$P(\exp_x(t\nu)) = x \text{ for all } t \geq 0$$

where $SN(S)$ denotes the unit normal bundle of S in M .

(B) For any geodesic $\gamma \subset S$ and any vector field $\nu \in \Gamma(SN(S))$ parallel along γ , the "horizontal" curves $\gamma_t, \gamma_t(u) = \exp_{\gamma(u)}(t\nu)$, are geodesics, filling a flat totally geodesic strip ($t \geq 0$). Moreover, if $\gamma[u_0, u_1]$ is minimizing, then all $\gamma_t[u_0, u_1]$ are also minimizing.

Perelman's proof of Soul Conjecture

The Soul Conjecture is now a corollary of (B). By lemma, the normal exponential map is surjective. Therefore we can find a normal geodesic from a soul S to the point p whose sectional curvature is always positive. If $\dim(S) > 0$, p will sit in a flat surface formed by geodesic strip, which contradicts the property of p .

Perelman's proof of Soul Conjecture

Proof.

We observe first that it is sufficient to show that if (A) and (B) hold for $0 \leq t \leq l$ for some $l \geq 0$, then they continue to hold for $0 \leq t \leq l + \epsilon(l)$, for some $\epsilon(l) > 0$.

Suppose that (A) and (B) hold for $0 \leq t \leq l$. For small $r \geq 0$, consider the function

$$f(r) = \max\{d(x, P(\exp_x(l+r)\nu)) : x \in S, \nu \in SN_x(S)\}$$

Then $f(0) = 0$. And we contend that f is a Lipschitz function.

Perelman's proof of Soul Conjecture

Actually, we have:

$$\begin{aligned}d(x, P(\exp_x(l + r_1)\nu)) \\ \leq d(x, P(\exp_x(l + r_2)\nu)) + d(P(\exp_x(l + r_2)\nu), P(\exp_x(l + r_1)\nu)) \\ \leq f(r_2) + d(\exp_x(l + r_2)\nu, \exp_x(l + r_1)\nu) \\ \leq f(r_2) + |r_1 - r_2|\end{aligned}$$

which implies $f(r_1) \leq f(r_2) + |r_1 - r_2|$. Similarly, we can prove the other side and show that f is Lipschitz continuous.

Perelman's proof of Soul Conjecture

We are going to show that the upper left derivative of f is nowhere positive for small r . Hence by lemma, f is non-increasing. Since $f \geq 0$ and $f(0) = 0$, f is identically zero for small r .

Fix $r > 0$. Let $f(r) = d(x_0, \bar{x}_0)$ where $\bar{x}_0 = P(\exp_{x_0}(l+r)\nu_0)$. Since r is small, and f is continuous, we may assume that $f(r) < \text{inrad}(S)$. Pick a point $x_1 \in S$ so that x_0 lies on a minimizing geodesic between \bar{x}_0 and x_1 ; let $x_0 = \gamma(u_0), x_1 = \gamma(u_1)$. Let $\nu \in \Gamma(SN(S))$ be a parallel vector field along γ , $\nu_{x_0} = \nu_0$. Then according to our assumption, the curve $\gamma_t(u) = \exp_{\gamma(u)}(t\nu), 0 \leq t \leq l$, are minimizing geodesics of filling a flat totally geodesic rectangle.

Perelman's proof of Soul Conjecture

It follows our assumption that:

- 1 The tangent vector field of normal geodesics is parallel along "horizontal" geodesic;
- 2 The tangent vector field of "horizontal" geodesics is parallel along normal geodesic.

In fact, denote by $F(t, u) = \exp_{\gamma(u)}(t\nu)$, we have:

$$\frac{D}{\partial t} \frac{D}{\partial u} \frac{\partial F}{\partial t} = \frac{D}{\partial u} \frac{D}{\partial t} \frac{\partial F}{\partial t} + R\left(\frac{\partial F}{\partial u}, \frac{\partial F}{\partial t}\right) \frac{\partial F}{\partial t} = 0$$

Hence

$$\frac{D}{\partial u} \frac{\partial F}{\partial t} = \frac{D}{\partial u} \frac{\partial F}{\partial t}(u, 0) = 0$$

which proves the first assertion. By torsion-free, the second holds.

Perelman's proof of Soul Conjecture

In particular, the normal geodesic and "horizontal" geodesic are orthogonal everywhere; and all "horizontal" geodesics are of constant length.

According to Berger's comparison theorem, the arcs of γ_{l+r} are no longer than corresponding arcs of γ_l , with equality if and only if γ_t , $l \leq t \leq l+r$, are geodesics filling a flat totally geodesic rectangle.

Perelman's proof of Soul Conjecture

Now consider the point $\bar{x}_1 = P(\exp_{\gamma(u_1)}(l+r)\nu)$. Using the distance decreasing property of P and the above observation we get:

$$\begin{aligned}d(\bar{x}_0, \bar{x}_1) &\leq d(\exp_{\gamma(u_0)}(l+r)\nu, \exp_{\gamma(u_1)}(l+r)\nu) \\ &\leq L(\gamma_{l+r}[u_0, u_1]) \\ &\leq L(\gamma_l[u_0, u_1]) = d(x_0, x_1)\end{aligned}$$

On the other hand,

$$d(x_1, \bar{x}_1) \leq f(r) = d(x_0, \bar{x}_0)$$

Taking into account that by construction

$$d(x_0, \bar{x}_0) + d(x_0, x_1) = d(\bar{x}_0, x_1) \leq d(x_1, \bar{x}_1) + d(\bar{x}_0, \bar{x}_1)$$

We see then the above must be equalities, and therefore

$$\gamma_t[u_0, u_1], \quad l \leq t \leq l + r$$

are minimizing geodesics filling a flat totally geodesic rectangle.

Perelman's proof of Soul Conjecture

Now let $\delta \rightarrow 0$, we obtain

$$\begin{aligned} f(r - \delta) &\geq d(x_1, P(\exp_{x_1}(l + r - \delta)\nu)) \\ &\geq d(\bar{x}_0, x_1) - d(\bar{x}_0, P(\exp_{x_1}(l + r - \delta)\nu)) \\ &\geq d(\bar{x}_0, x_1) - d(\exp_{x_1}(l + r - \delta)\nu, \exp_{x_0}(l + r)\nu) \end{aligned}$$

By Toponogov's comparison theorem,

$$\begin{aligned} &d(\exp_{x_1}(l + r - \delta)\nu, \exp_{x_0}(l + r)\nu) \\ &\leq \sqrt{d(\exp_{x_1}(l + r)\nu, \exp_{x_0}(l + r)\nu) + \delta^2} \\ &= d(\exp_{x_1}(l + r)\nu, \exp_{x_0}(l + r)\nu) + O(\delta^2) \end{aligned}$$

Perelman's proof of Soul Conjecture

Hence we have:

$$\begin{aligned} f(r - \delta) &\geq d(\bar{x}_0, x_1) - d(\exp_{x_1}(l + r)\nu, \exp_{x_0}(l + r)\nu) - O(\delta^2) \\ &= d(\bar{x}_0, x_1) - d(x_0, x_1) - O(\delta^2) \\ &= d(\bar{x}_0, x_0) - O(\delta^2) = f(r) - O(\delta^2) \end{aligned}$$

which shows that the upper left derivative of f is nowhere positive, whence $f(r) \equiv 0$ for $0 \leq r \leq \epsilon(l)$. Therefore, (A) is proved for $0 \leq r \leq \epsilon(l)$. To prove (B) for such t , one can repeat a part of argument above, taking into account that $(x_0, \nu_0), \gamma, x_1$ can now be chosen arbitrarily since $\bar{x}_0 = x_0, \bar{x}_1 = x_1$. **Q.E.D**

- 1 All souls of a given Riemannian manifold are isometric; the set of M 's all souls is named as "Heaven".
- 2 Perelman also proved that such a retraction is a Riemannian submersion of class C^1 and gave an upper bound of second fundamental forms of its fibers.
- 3 In the end of his paper, he addressed an open question for an Alexandrov space version of this theorem, which may occur as Gromov-Hausdorff limits of blowups of Riemannian manifolds, collapsing with lower bound on sectional curvature.

- 1 G.Perelman,
Proof of the soul conjecture of Cheeger and Gromoll, Journal of Differential Geometry.40(1994)209-212
- 2 Jeff Cheeger and Detlef Gromoll,
On the structure of complete manifolds of nonnegative curvature, Annals of Mathematics, Second Series, Vol.96, No.3(Nov.,1972),pp.413-443

Thank you!